AN ELEMENTARY INVARIANT PROBLEM
AND GENERAL LINEAR GROUP COHOMOLOGY
RESTRICTED TO THE DIAGONAL SUBGROUP

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ABSTRACT. Conjecturally, for p an odd prime and R a certain ring of p-integers, the stable general linear group $GL(R)$ and the étale model for its classifying space have isomorphic mod p cohomology rings. In particular, these two cohomology rings should have the same image with respect to the restriction map to the diagonal subgroup. We show that a strong unstable version of this last property holds for any rank if p is regular and certain homology classes for $SL_2(R)$ vanish. We check that this criterion is satisfied for $p = 3$ as evidence for the conjecture.

1. INTRODUCTION

Let p be a prime and $R = \mathbb{Z}[\zeta, 1/p]$ the ring of p-integers containing a primitive p-th root of unity ζ. A great deal of interest has been generated by the mod p cohomology of the stable general general linear group $GL(R)$. Dwyer and Friedlander [4] constructed a topological space $X(R)$ based on the étale topological type of the scheme $Spec(R)$ and a continuous map

$$f : BGL(R) \to X(R)$$

where $BGL(R)$ is the classifying space of the discrete group $GL(R)$. They showed that f is injective on mod p cohomology if p is regular [5, 6.4]. Voevodsky [9] proved a conjecture of Milnor having as consequence the fact that f is a p-equivalence (i.e., induces an isomorphism on mod p cohomology) for $p = 2$. Quillen [8] and Lichtenbaum [6] have studied a conjectural relationship between the algebraic K-theory of R and the values of an associated zeta function. Their work suggests that f should be a p-equivalence for any prime p.

In particular, let $D(R)$ be the subgroup of diagonal matrices inside $GL(R)$ and consider the following diagram on mod p cohomology $H^* = H^*(-, \mathbb{F}_p)$:

$$H^*X(R) \xrightarrow{f^*} H^*BGL(R) \xrightarrow{res} H^*BD(R)$$

where f^* is induced by f and res by the inclusion $D(R) \subset GL(R)$. If f is a p-equivalence, then res and $res \circ f^*$ have the same image (see also Remark 4.11).

In this article we show that a strong unstable version of the last property holds under certain conditions. Namely, let n be a nonnegative integer and $GL_n(R)$ the

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general linear group of rank n over R. Then we can replace GL, D, X, f, res above respectively by GL_n, D_n, X_n, f_n, res_n and keep everything else unchanged.

Theorem 1.1. *Let p be an odd regular prime, M_n the image of $res_n \circ f_n^*$, and I_n the image of res_n. If $M_2 = I_2$, then $M_n = I_n$ for all $n \geq 0$.*

We can actually strengthen this theorem by weakening the condition $M_2 = I_2$ and requiring instead that certain homology classes for $SL_2(R)$ vanish (see Theorem 4.10). As an example, when $M_2 = I_2$ holds we can cite our previous result (see also Remark 4.4).

Corollary 1.2 ([1, 1.3]). *If $p = 3$, then $M_n = I_n$ for all $n \geq 0$.*

Theorem 1.1 was inspired by [3] where a similar statement was proven in case $p = 2$. Our generalization is based on two observations: (i) the mod p cohomology ring of the étale model $X_n(R)$ has a particularly simple structure for p odd and regular [5] and (ii) the maps dual to res_n assemble for $n \geq 0$ into a ring homomorphism on homology with respect to the ring structure induced by matrix block-multiplication $GL_n(R) \times GL_m(R) \to GL_{n+m}(R)$. Based on this last ring structure, we formulate and solve a general invariant problem (Theorem 3.4) having as consequence Theorem 1.1. Surprisingly, the solution to the invariant problem is elementary and purely combinatorial, although quite tricky.

The article is organized as follows. In §2 we review the mod p cohomology ring of the étale model $X_n(R)$. In §3 we formulate and solve the invariant problem and deduce Theorem 1.1. In §4 we strengthen and generalize our theorem.

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2. Etale model cohomology

In what follows we assume that p is an odd regular prime and keep the notation from the Introduction. In [5, 6.3] the graded ring associated to a filtration of $H^*X_n(R)$ was computed and we review this calculation here from a slightly different perspective:

Theorem 2.1. *Let p be an odd regular prime and n a nonnegative integer. Then*

$$H^*X_n(R) \approx \mathbb{F}_p[c_1, c_2, ..., c_n] \otimes \bigotimes_{i=1}^{r} \Lambda(e_{i,1}, e_{i,2}, ..., e_{i,n})$$

where $r = (p+1)/2$, $\otimes$ is the tensor product over $\mathbb{F}_p$, and Λ is the symbol for an exterior $\mathbb{F}_p$-algebra. For $1 \leq j \leq n$ and $1 \leq i \leq r$, c_j is the Chern class of degree $|c_j| = 2j$ associated with a complex embedding of R and $e_{i,j}$ is a class of degree $|e_{i,j}| = 2j - 1$.

The following corollary is essentially [5, 6.2] in a form more suitable for the purpose of this article.

Corollary 2.2. *With the same notation and hypotheses as in Theorem 2.1,*

$$H^*BD_n(R) \approx \mathbb{F}_p[x_1, x_2, ..., x_n] \otimes \bigotimes_{i=1}^{r} \Lambda(y_{i,1}, y_{i,2}, ..., y_{i,n})$$

where the degrees of the generators are given by $|x_j| = 2$ *and* $|y_{i,j}| = 1$ *and the map* $res_n \circ f_n^*$ *is a ring monomorphism sending* c_j *to the symmetrization of* $x_1...x_j$ *and* $e_{i,j}$ *to the symmetrization of* $x_1...x_{j-1}y_{i,j}$ *for* $1 \le j \le n$ *and* $1 \le i \le r$.

Remark 2.3. For $n \ge 0$, let Σ_n be the symmetric group of $\{1, 2, \ldots, n\}$. Then Σ_n acts in an obvious manner on the diagonal subgroup $D_n(R)$. This action induces an action of Σ_n on $H^*D_n(R)$ via ring homomorphims $x \mapsto x^\sigma$ sending x_j to $x_{\sigma(j)}$ and $y_{i,j}$ to $y_{i,\sigma(j)}$ for $1 \le i \le r$ and $1 \le j \le n$ where $\sigma \in \Sigma_n$. Then the symmetrization of an element $x \in H^*D_n(R)$ is the sum without repetitions of all the elements x^σ with $\sigma \in \Sigma_n$, i.e., the sum $\sum x^\sigma$ over the orbit of x.

2.1. **Etale models.** Before sketching the proof of Theorem 1.1 and its corollary we review some ingredients. Let n be a nonnegative integer, p a prime, and $R_0 = \mathbb{Z}[1/p]$. Then the *étale model* $X_n(S)$ *at* p is a topological space naturally assigned to each Noetherian R_0-algebra S and the map

$$f_n : BGL_n(S) \to X_n(S)$$

is a natural transformation [4].

Proposition 2.4 ([1, 2.3], [5, 2.8, 2.11])**.** *The following properties hold for any prime p and nonnegative integer n:*

(1) *If S is a complete local ring with residue field k of characteristic different from p, then the induced map $X_n(S) \to X_n(k)$ is a p-equivalence.*
(2) *If S is a finite field of order prime to p, then f_n is a p-equivalence.*
(3) *If S is the field of complex numbers, then $X_n(S)$ is p-equivalent to the classifying space $BGL_n^{top}(S)$ of the Lie group $GL_n^{top}(S)$.*

We recall that the cohomology ring of $BGL_n^{top}(\mathbb{C})$, the infinite complex Grassmannian, is a polynomial ring in the universal Chern classes c_1, c_2,..., c_n. Also, if $\mathfrak{q}_i$ is a prime ideal of R with the residue field k_i of order $\equiv 1 \mod p$ and $\mathfrak{q}_i$-adic completion $\hat{R}_{\mathfrak{q}_i}$, then the cohomology ring of $BGL_n(k_i)$ was calculated by Quillen in [8]. As a corollary of these known facts and Proposition 2.4 we have:
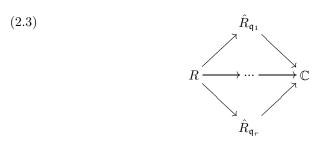
$$(2.1) \qquad H^*X_n(\mathbb{C}) \approx H^*BGL_n^{top}(\mathbb{C}) \approx \mathbb{F}_p[c_1, c_2, ..., c_n],$$

$$(2.2) \qquad H^*X_n(\hat{R}_{\mathfrak{q}_i}) \approx H^*BGL_n(k_i) \approx \mathbb{F}_p[c_1, c_2, ..., c_n] \otimes \Lambda(e_{i,1}, e_{i,2}, ..., e_{i,n})$$

where $|c_j| = 2j$ and $|e_{i,j}| = 2j - 1$ for $1 \le j \le n$.

2.2. **Proofs.** Now we are ready to sketch the proofs of Theorem 2.1 and Corollary 2.2.

Proof of Theorem 2.1. According to [2] there are $r = (p + 1)/2$ prime ideals of R, say $\mathfrak{q}_i$ for $1 \le i \le r$, and a commutative diagram in the category of Noetherian

R_0-algebras

(2.3)

such that $X_n((2.3))$ is a homotopy cartesian diagram in the category of topological spaces. Moreover, the residue field of $\mathfrak{q}_i$, say k_i, has order $\equiv 1 \mod p$ for $1 \leq i \leq r$. In particular, the formula (2.2) holds for all $1 \leq i \leq r$ and combining these formulae with (2.1), we deduce that the Eilenberg-Moore spectral sequence associated with the homotopy cartesian diagram $X_n((2.3))$ collapses to a ring isomorphism

$$H^* X_n(R) \approx \bigotimes_{i=1}^{r} H^* X_n(\hat{R}_{\mathfrak{q}_i}).$$

Here the tensor product is over $H^* X_n(\mathbb{C})$ and hence the theorem follows from (2.2) and (2.1).   □

Proof of Corollary 2.2. By Dirichlet's Unit Theorem, $D_1(R)$ is a product of $r-1$ copies of the integers $\mathbb{Z}$ where $r = (p+1)/2$ and a finite cyclic group C with its p-primary part generated by the primitive p-th root of unity ζ. By the Künneth theorem, it follows that $H^* BD_n(R)$ is a tensor product of $n(r-1)$ copies of $H^*(\mathbb{Z})$ and n copies of $H^*(C)$ and therefore has the desired ring structure by standard calculations.

In order to prove the second part of the corollary, we can apply the same method as in the proof of Theorem 2.1. With the same notation, the restriction maps

$$H^* BGL_n(k_i) \to H^* BD_n(k_i), \quad 1 \leq i \leq r$$

can be identified according to [8, p. 564] with the symmetrization maps

$$s_i : \mathbb{F}_p[c_1, ..., c_n] \otimes \Lambda(e_{i,1}, ..., e_{i,n}) \to \mathbb{F}_p[x_1, ..., x_n] \otimes \Lambda(y_{i,1}, ..., y_{i,n}).$$

Let us take the tensor product of all s_i, $1 \leq i \leq r$, with respect to $\mathbb{F}_p[c_1, c_2, ..., c_n]$ in the source and $\mathbb{F}_p[x_1, x_2, ..., x_n]$ in the target. Then we can identify the source of $\bigotimes_{i=1}^{r} s_i$ with $H^* X_n(R)$ by Theorem 2.1, the target with $H^* BD_n(R)$ by the first part of the corollary, and hence, the map $\bigotimes_{i=1}^{r} s_i$ itself with $res_n \circ f_n^*$ by naturality. The injectivity of $\bigotimes_{i=1}^{r} s_i$ can be shown as in [8, Lemma 9].   □

3. An invariant problem

In this section, we deduce Theorem 1.1 from a more general invariant problem, which we formulate and solve after some preliminary steps.

3.1. **Preliminary steps.** In order to prove Theorem 1.1 we need to use the matrix block-multiplication $GL_m(R) \times GL_n(R) \to GL_{m+n}(R)$ for $m \geq 0$ and $n \geq 0$ (see

Remark 3.5). This multiplication induces a mod p homology ring structure on each of the direct sums appearing in the following diagram (compare to [3]):

$$\bigoplus_{n=0}^{\infty} H_* BD_n(R) \xrightarrow{\oplus res_{n*}} \bigoplus_{n=0}^{\infty} H_* BGL_n(R) \xrightarrow{\oplus f_{n*}} \bigoplus_{n=0}^{\infty} H_* X_n(R)$$

where res_{n*} is dual to res_n and f_{n*} is dual to f_n^* for $n \geq 0$ (the homology and cohomology are dual as vector spaces over $\mathbb{F}_p$). With respect to this ring structure the maps $\bigoplus_{n=0}^{\infty} res_{n*}$ and $\bigoplus_{n=0}^{\infty} f_{n*}$ are ring homomorphisms. This diagram induces a diagram of rings and ring homomorphisms

$$(3.1) \qquad \bigoplus_{n=0}^{\infty} H_* BD_n(R) \xrightarrow{\rho} \bigoplus_{n=0}^{\infty} I_n' \to \bigoplus_{n=0}^{\infty} M_n'$$

where V' denotes the degreewise dual of the graded vector space V over $\mathbb{F}_p$ and the maps in the diagram are direct sums of maps dual to the inclusions $M_n \subseteq I_n$ and $I_n \subseteq H^* BD_n(R)$ for $n \geq 0$ (same notation as in Theorem 1.1). These properties suggest an invariant problem, which we formulate in the next subsection.

3.2. Formulating the invariant problem. Let r be a nonnegative integer and F a field of characteristic different from 2. Both r and F are fixed throughout the discussion. For each $n \geq 0$ we define the following graded algebra over F:

$$(3.2) \qquad A_n = F[x_1, x_2, ..., x_n] \otimes \bigotimes_{i=1}^{r} \Lambda(y_{i,1}, y_{i,2}, ..., y_{i,n})$$

with $|x_j| = 2$ and $|y_{i,j}| = 1$ for $1 \leq i \leq r$ and $1 \leq j \leq n$. The symmetric group Σ_n acts on A_n as described in Remark 2.3. With respect to this action we define the following subring of invariants:

$$(3.3) \qquad E_n = F[c_1, c_2, ..., c_n] \otimes \bigotimes_{i=1}^{r} \Lambda(e_{i,1}, e_{i,2}, ..., e_{i,n})$$

where c_j and $e_{i,j}$ are the symmetrizations of $x_1...x_j$ and $x_1...x_{j-1}y_{i,j}$ for $1 \leq j \leq n$ and $1 \leq i \leq r$.

Let V' denote the degreewise dual of the graded vector space V over F and define an algebra structure on the direct sum $\bigoplus_{n=0}^{\infty} A_n'$ as follows. For each $n \geq 0$, let $S(n)$ be the set of all sequences

$$(3.4) \qquad I = (k_1, ..., k_n; \epsilon_{1,1}, ..., \epsilon_{1,n}; ...; \epsilon_{r,1}, ..., \epsilon_{r,n})$$

of nonnegative integers with $\epsilon_{i,j} \in \{0, 1\}$ for $1 \leq i \leq r$ and $1 \leq j \leq n$. The monomials

$$x^I = x_1^{k_1}...x_n^{k_n} \prod_{i=1}^{r} y_{i,1}^{\epsilon_{i,1}}...y_{i,n}^{\epsilon_{i,n}}$$

with $I \in S(n)$ as in (3.4) form an additive basis for A_n; let (u_I) be the dual basis in A_n'.

Definition 3.1. If $I \in S(n)$ is of the form (3.4) and $J \in S(m)$ is of the form

$$J = (l_1, ..., l_m; \phi_{1,1}, ..., \phi_{1,m}; ...; \phi_{r,1}, ..., \phi_{r,m}),$$

then we define the concatenation $IJ \in S(n+m)$ by:

$$IJ = (k_1, ..., k_n, l_1, ..., l_m; \epsilon_{1,1}, ..., \epsilon_{1,n}, \phi_{1,1}, ..., \phi_{1,m}; ...; \epsilon_{r,1}, ..., \epsilon_{r,n}, \phi_{r,1}, ..., \phi_{r,m}).$$

Definition 3.2. We define the product of any two basis elements u_I and u_J by the following equations:

$$u_I u_J = \epsilon u_{IJ}, \qquad x^{IJ} = \epsilon x^{IO} x^{OJ}$$

where $I \in S(n)$, $J \in S(m)$, O stands for sequences of the form $(0, 0, ..., 0)$ such that both IO and OJ belong to $S(n+m)$, and $\epsilon \in \{-1, 1\}$ is chosen such that the second equation is an identity in A_{n+m}.

Lemma 3.3. *With respect to the product defined in 3.2, the direct sum $\bigoplus_{n=0}^{\infty} A'_n$ is an algebra isomorphic to the tensor algebra generated by A'_1.*

Proof. Let θ be the linear map from the tensor algebra generated by A'_1 which sends each basis element of the form $u_{I_1} \otimes u_{I_2} \otimes ... \otimes u_{I_n}$ to ϵu_I where $n \geq 0$, $I_k \in S(1)$ for $1 \leq k \leq n$, $I = I_1 I_2 ... I_n$, and $\epsilon \in \{-1, 1\}$ is chosen such that the following equation

$$x^I = \epsilon x^{I_1 O ... O} x^{O I_2 O ... O} x^{O ... O I_n}$$

is an identity in A_n. Here $O \in S(1)$ is a sequence of the form $(0, 0, ..., 0)$ occurring $n - 1$ times in each exponent. Then θ is the desired algebra isomorphism. $\qquad\square$

Now we can formulate the following invariant problem:

Theorem 3.4. *For each $n \geq 0$, let V_n be a graded linear subspace of A_n that contains E_n and assume that there is a ring structure on the direct sum $\bigoplus_{n=0}^{\infty} V'_n$ such that the linear map*

$$g = \bigoplus_{n=0}^{\infty} g_n : \bigoplus_{n=0}^{\infty} A'_n \to \bigoplus_{n=0}^{\infty} V'_n$$

is a ring homomorphism with respect to Definition 3.2 where g_n is dual to the inclusion $V_n \subseteq A_n$ for each $n \geq 0$. Then $E_2 = V_2$ implies $E_n = V_n$ for all $n \geq 0$.

Remark 3.5. Let us drop the hypothesis that g is a ring homomorphism and assume instead that V_n is a subring of the invariants $A_n^{\Sigma_n}$ that contains E_n and whose image with respect to the map which kills all generators x_j and $y_{i,j}$ for $j > 2$ (and keeps the others) becomes the subring $E_2 \subseteq A_2$. It is a pleasant warm-up exercise to show that in case $r = 0$ and $r = 1$ the invariants (and therefore V_n) agree with the ring E_n. However, for $r > 1$ the invariants are bigger and it is not true that $E_n = V_n$. For instance, let $n \geq 3$ and let y_n be the symmetrization of the monomial $y_{1,1} y_{1,2} y_{1,3} y_{2,1} y_{2,2} y_{2,3}$ in A_n. Then the subring $V_n = E_n[y_n]$ generated by E_n and y_n is a counterexample.

3.3. **The proof of Theorem 1.1.** We assume the notation and hypotheses of 1.1. By 2.1 and 2.2 the rings $H^* BD_n(R)$ and M_n can be identified respectively with the rings A_n and E_n as defined in (3.2) and (3.3) for $F = \mathbb{F}_p$, $r = (p+1)/2$, and $n \geq 0$. With these identifications, I_n is a graded linear subspace of A_n that contains E_n for $n \geq 0$. Therefore, the map ρ from (3.1) can be identified with the map g from Theorem 3.4 for $V_n = I_n$. According to our discussion in §3.1, the map ρ is a ring homomorphism with respect to the algebra structure on $\bigoplus_{n=0}^{\infty} A'_n$ induced from the matrix block-multiplication. By the Künneth theorem on homology, this algebra structure agrees with the algebra structure defined in 3.2 (compare to Lemma 3.3). Hence, Theorem 1.1 follows from Theorem 3.4.

3.4. **The proof of Theorem 3.4.** We assume now the notation of §3.2. Let us compose the map g with the linear map

$$h = \bigoplus_{n=0}^{\infty} h_n : \bigoplus_{n=0}^{\infty} V_n' \to \bigoplus_{n=0}^{\infty} E_n'$$

where h_n is dual to the inclusion $E_n \subseteq V_n$ for each $n \geq 0$. Let $t = hg$ and $t_n = h_n g_n$ for $n \geq 0$. Then we show that

Proposition 3.6. $\ker(t)$ *is an ideal generated by* $\ker(t_2)$.

Assuming this proposition and that g is a ring homomorphism, then $\ker(g)$ is an ideal contained in $\ker(t)$ and hence, $E_2 = V_2$ will imply that $\ker(g) = \ker(t)$. Because g and t are epimorphisms as direct sums of maps dual to inclusions, it follows that h is an isomorphism. Therefore, $E_n = V_n$ for all $n \geq 0$.

Remark 3.7. A particular case of Proposition 3.6 appeared as [1, Lemma 3.3] but the proof given in [1, p.12] is obviously inaccurate.

3.5. **The proof of Proposition 3.6.** The strategy we are pursuing in proving 3.6 is the following. We first study the map t in terms of additive bases and show that $\ker(t)$ is an ideal. Then we define a map $\alpha : S(n) \to S(n)$ which controls the way t maps basis elements. By using α, we show that if $\ker(t)$ is not generated by $\ker(t_2)$, then we have an infinite ascent phenomenon inside a certain subset S of $S(n)$. Because S will be finite, our assumption is false, proving Proposition 3.6.

Let us start with an additive basis for E_n given by the monomials

$$c^I = c_1^{k_1}...c_n^{k_n} \prod_{i=1}^{r} e_{i,1}^{\epsilon_{i,1}}...e_{i,n}^{\epsilon_{i,n}}$$

where $I \in S(n)$ is of the form (3.4) and let (v_I) be the dual basis in E_n'. Then we have the following duality equations:

$$(3.5) \qquad c^K \quad = \quad \sum_{I \in S(n)} [K:I]x^I \text{ for } K \in S(n),$$

$$(3.6) \qquad t(u_I) \quad = \quad \sum_{K \in S(n)} [K:I]v_K \text{ for } I \in S(n)$$

where the coefficients $[K:I] \in F$ are zero almost everywhere in each sum.

Lemma 3.8. *There is a ring structure on the direct sum* $\bigoplus_{n=0}^{\infty} E_n'$ *such that* t *is a ring homomorphism with respect to Definition 3.2.*

Proof. Let us define a ring structure on the direct sum $\bigoplus_{n=0}^{\infty} E_n'$ by the following equations which are similar to 3.2:

$$v_K v_L = \delta v_{KL}, \quad c^{KL} = \delta c^{KO} c^{OL}$$

where $K \in S(n)$, $L \in S(m)$, and $\delta \in \{-1, 1\}$ is chosen such that the second equation is an identity in E_{n+m}. Then we can use (3.5), (3.6), and 3.2 to show that t is a ring homomorphism. $\square$

Next, we need a way to bound indices I by natural numbers (see 3.9). Also, we use the following lexicographical order: two sequences of integers $a = (a_1, ..., a_m)$ and $a' = (a_1', ..., a_m')$ of the same length $m \geq 0$ are in the relation $a > a'$ if $a_1 > a_1'$ or there is $1 \leq s < m$ such that $a_j = a_j'$ for $1 \leq j \leq s$ and $a_{s+1} > a_{s+1}'$.

Definition 3.9. Let $n \geq 0$ and let $I \in S(n)$ be of the form (3.4). Then we define the *degree* of I by

$$deg(I) = \sum_{j=1}^{n} (k_j + \sum_{i=1}^{r} \epsilon_{i,j}).$$

Lemma 3.10. *Let $n \geq 0$. Then there is an injective map $\alpha : S(n) \to S(n)$ such that for any $I \in S(n)$ and $K \in S(n)$ the following conditions hold:*

(1) $[K : I] = 1$ *if $I = \alpha(K)$.*
(2) $[K : I] = 0$ *if $I > \alpha(K)$.*
(3) *If I is of the form (3.4), then I is in the image of α if and only if*

$$k_j - k_{j+1} \geq \sum_{i=1}^{r} \epsilon_{i,j+1} \text{ for all } 1 \leq j < n.$$

(4) $[K : I] = 0$ *if $deg(I) \neq deg(\alpha(K))$.*

Proof. Let us assume that $I \in S(n)$ has the form (3.4) and K the form

$$K = (a_1, ..., a_n; \phi_{1,1}, ..., \phi_{1,n}; ...; \phi_{r,1}, ..., \phi_{r,n}).$$

Then x^I is lexicographically with respect to I the leading term of the expansion of c^K as in (3.5) if and only if:

$$k_j = a_j + ... + a_n + \sum_{i=1}^{r} \phi_{i,j+1} + ... + \sum_{i=1}^{r} \phi_{i,n} \quad \text{for } 1 \leq j < n,$$
$$k_n = a_n,$$
$$\epsilon_{i,j} = \phi_{i,j} \quad \text{for } 1 \leq j \leq n \text{ and } 1 \leq i \leq r.$$

Given $I \in S(n)$, this system of equations has a solution $K \in S(n)$ if and only if I satisfies the condition (3) of the lemma. Moreover, if this condition is satisfied, then the solution K is unique. Hence, if we define $\alpha(K)$ to be the only element in $S(n)$ such that $x^{\alpha(K)}$ is the leading term of c^K, then the map α is injective and satisfies (1) to (3). In order to prove (4) we only need to observe that all monomials x^I occurring with nonzero coefficients in the expansion of c^K have the same value for $deg(I)$ and $x^{\alpha(K)}$ is one of these monomials. $\square$

With these preparations and notation, we can prove Proposition 3.6 as follows. By Lemma 3.8, t is a ring homomorphism and therefore, $\ker(t)$ contains the ideal generated by $\ker(t_2)$, denoted $(\ker(t_2))$. Now, assume that $\ker(t)$ is bigger than $(\ker(t_2))$. For each $n \geq 0$ and $d \geq 0$, let M be the set of all elements $x \in A'_n$ such that $x \in \ker(t)$, $x \notin (\ker(t_2))$, and x can be written in the form

$$(3.7) \qquad\qquad x = \sum_{I \in S(n), \ deg(I)=d} (x : I) u_I$$

where we define $(x : I) \in F$ for all $I \in S(n)$ by setting $(x : I) = 0$ if $deg(I) \neq d$. Let us fix n and d such that the set M is not empty (we can do this according to our assumption, (3.6), and Lemma 3.10 (4)). Now, let S be the set of all $I \in S(n)$ satisfying the following conditions: $deg(I) = d$ and there is $x \in M$ such that $(x : J) = 0$ for all $J < I$. This set is finite and not empty. Let $I_0 \in S$ be lexicographically the maximal element of S.

We claim that we can find an element $I'_0 \in S$ which contradicts the maximality property of I_0. More specifically, we have the following

Proposition 3.11. *Let $x_0 \in M$ such that $(x_0 : I) = 0$ for all $I < I_0$. Then there is $u \in (\ker(t_2))$ such that u can be written in the form (3.7) with $(u : I_0) = (x_0 : I_0)$ and $(u : I) = 0$ for $I < I_0$.*

Now, the element $x_0' = x_0 - u$ belongs to M and there is $I_0' \in S(n)$ such that $deg(I_0') = d$ and $(x_0' : I) = 0$ for all $I < I_0'$ but $(x_0' : I_0') \neq 0$. Therefore, $I_0' \in S$ and by the properties of u and x_0, $(x_0' : I) = 0$ for all $I \leq I_0$. It follows that $I_0' > I_0$, proving the claim and Proposition 3.6.

3.6. **The proof of Proposition 3.11.** We need to make use of the map α and its properties given in Lemma 3.10.

Case 1: Assume that there is $K \in S(n)$ such that $I_0 = \alpha(K)$. By 3.10 (1) (2) we have $[K : I_0] = 1$ and $[K : I] = 0$ for all $I > I_0$. Combining these with (3.6) and the properties of x_0 and I_0, we have

$$0 = \left\langle c^K, t(x_0) \right\rangle = \sum_{I \in S(n),\ deg(I)=d} [K : I](x_0 : I) = (x_0 : I_0)$$

where $\langle\ ,\ \rangle$ is the duality pairing. Hence, $u = 0$ has the desired properties.

Case 2: Assume that there is no $K \in S(n)$ such that $I_0 = \alpha(K)$. Then by 3.10 (3) there is $1 \leq j < n$ such that $I_0 = I_1 I_2 I_3$ where $I_1 \in S(j - 1)$, $I_2 \in S(2)$, and $I_3 \in S(n - j - 1)$ are of the form

$$I_1 = (k_1, ..., k_{j-1}; \epsilon_{1,1}, ..., \epsilon_{1,j-1}; ...; \epsilon_{r,1}, ..., \epsilon_{r,j-1}),$$

$$I_2 = (k_j, k_{j+1}; \epsilon_{1,j}, \epsilon_{1,j+1}; ...; \epsilon_{r,j}, \epsilon_{r,j+1}) \text{ with } k_j - k_{j+1} < \sum_{i=1}^{r} \epsilon_{i,j+1},$$

$$I_3 = (k_{j+2}, ..., k_n; \epsilon_{1,j+2}, ..., \epsilon_{1,n}; ...; \epsilon_{r,j+2}, ..., \epsilon_{r,n}).$$

Subcase 2(a): Assume that $t(u_{I_2}) = 0$. Then $u_{I_0} = \pm u_{I_1} u_{I_2} u_{I_3} \in (\ker(t_2))$ and $u = (x_0 : I_0)u_{I_0}$ has the desired properties.

Subcase 2(b): Assume that $t(u_{I_2}) \neq 0$. Let $d' = deg(I_2)$ and define T to be the set of all $K \in S(2)$ such that $deg(\alpha(K)) = d'$. Because α is injective, T is a set of finite order, say m. Let us label the elements of T by $K_1, K_2,..., K_m$ such that $\alpha(K_1) < \alpha(K_2) < ... < \alpha(K_m)$.

Lemma 3.12. *There is $1 \leq s \leq m$ such that $I_2 < \alpha(K_s)$ and $[K_i : I_2] = 0$ for all $1 \leq i < s$.*

Proof. By (3.6) and 3.10 (4), we have

$$t(u_{I_2}) = \sum_{i=1}^{m} [K_i : I_2] v_{K_i}.$$

Because $t(u_{I_2}) \neq 0$, there is $1 \leq s \leq m$ such that $[K_s : I_2] \neq 0$ and $[K_i : I_2] = 0$ for $1 \leq i < s$. By 3.10 (3) there is no $K \in S(2)$ such that $I_2 = \alpha(K)$ and combining this with 3.10 (2), we get $I_2 < \alpha(K_s)$. $\square$

Corollary 3.13. *With s as in Lemma 3.12, the following system of equations*

$$0 = [K_j : I_2] + \sum_{i=s}^{m} [K_j : \alpha(K_i)]\lambda_i \text{ for all } 1 \leq j \leq m$$

has a solution $\lambda_s, \lambda_{s+1},..., \lambda_m \in F$.

Proof. By 3.10 (1) (2), we have $[K_i : \alpha(K_i)] = 1$ for all $s \le i \le m$ and $[K_j : \alpha(K_i)] = 0$ for all $1 \le j < i \le m$. Combining these with 3.12, we deduce that the system has indeed a solution. $\qquad\square$

Let $\lambda_i \in F$ for $s \le i \le m$ be a solution of the system in Corollary 3.13 and define

$$w = u_{I_2} + \sum_{i=s}^{m} \lambda_i u_{\alpha(K_i)}.$$

Then we have

$$
\begin{aligned}
t(w) &= \sum_{j=1}^{m} ([K_j : I_2] + \sum_{i=s}^{m} [K_j : \alpha(K_i)]\lambda_i) v_{K_j} \quad \text{by (3.6) and 3.10 (4)} \\
&= 0 \quad \text{by 3.13.}
\end{aligned}
$$

Therefore, $w \in \ker(t_2)$ and $u = \pm(x_0 : I_0) u_{I_1} w u_{I_3}$ has the desired properties, proving Proposition 3.11.

4. Etale obstruction classes

In this section we keep the hypotheses and notation from §2 and show that the condition $M_2 = I_2$ in Theorem 1.1 is satisfied if certain homology classes vanish.

Namely, let $SL_2(R)$ be the special linear group of 2×2 matrices over R with determinant 1 and consider the following map:

$$(4.1) \qquad \iota : D_1(R) \to SL_2(R), \quad \iota(a) = \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \text{ for } a \in D_1(R).$$

By Corollary 2.2, the mod p group cohomology $H^*D_1(R)$ can be identified with the ring

$$(4.2) \qquad \mathbb{F}_p[x] \otimes \bigotimes_{i=1}^{r} \Lambda(y_i), \ r = \frac{p+1}{2}, \ |x| = 2, \ |y_i| = 1 \text{ for } 1 \le i \le r.$$

For each $I \in S(1)$ of the form $I = (k, \epsilon_1, \epsilon_2, ..., \epsilon_r)$ as in §3.2, let

$$x^I = x^k \prod_{i=1}^{r} y_i^{\epsilon_i} \quad \text{and} \quad a(I) = \frac{1}{2}(\epsilon_1 + \epsilon_2 + ... + \epsilon_r - k)$$

where we recall that $\epsilon_i \in \{0, 1\}$ for $1 \le i \le r$. Then $H^*D_1(R)$ has an additive basis given by the monomials x^I with $I \in S(1)$. Let (u_I) be the dual basis in $H_*D_1(R)$. Then the map ι_* induced by ι on mod p homology sends each basis element u_I with $I \in S(1)$ to a homology class $\iota_*(u_I)$ in $H_*SL_2(R)$.

Definition 4.1. Let β be the Bockstein derivation on mod p cohomology. We call an identification of $H^*D_1(R)$ with the ring (4.2) *admissible* if $x \in \beta H^1 D_1(R)$.

Definition 4.2. Given an admissible identification, we say that $\iota_*(u_I) \in H_*SL_2(R)$ where $I \in S(1)$ is an *étale obstruction class* if $a(I)$ is a positive integer.

Proposition 4.3. *Let p be an odd regular prime and fix an admissible identification of $H^*D_1(R)$ with (4.2). Then the linear subspace of $H_*SL_2(R)$ spanned by the étale obstruction classes is trivial if and only if the condition $M_2 = I_2$ in Theorem 1.1 is satisfied.*

Remark 4.4. For $p = 3$, there is only one sequence $I \in S(1)$ satisfying 4.2, namely $I = (0, 1, 1)$. Therefore, given an admissible identification, there is only one obstruction class living in $H_2 SL_2(R)$. By [1, (5.1)], this homology group is zero, reproving 1.2 as a consequence of 1.1 and 4.3. For $p > 3$, it is an open problem whether $M_2 = I_2$ or not.

4.1. **The proof of Proposition 4.3.** The idea is to use the following commutative diagram of groups and group homomorphisms:

$$
\begin{array}{ccc}
D_1(R) \times D_1(R) & \xrightarrow{\iota \times Id} & SL_2(R) \times D_1(R) \\
\downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \pi} \\
D_2(R) & \xrightarrow{\quad incl. \quad} & GL_2(R)
\end{array}
$$

where ι is the map (4.1), Id is the identity map of $D_1(R)$, $incl.$ is the inclusion map $D_2(R) \subset GL_2(R)$, and the other two maps are defined by the formulae

$$
\tau(a, b) = \iota(a) \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix},
$$

$$
\pi(A, b) = A \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}
$$

for $a, b \in D_1(R)$ and $A \in SL_2(R)$ (on the right-hand side we use matrix multiplication). This diagram induces a commutative diagram on mod p cohomology

$$
(4.3) \qquad
\begin{array}{ccc}
H^* D_1(R) \otimes H^* D_1(R) & \xleftarrow{\iota^* \otimes Id} & H^* SL_2(R) \otimes H^* D_1(R) \\
\uparrow{\scriptstyle \tau^*} & & \uparrow{\scriptstyle \pi^*} \\
H^* D_2(R) \xleftarrow{\quad res_2 \quad} & H^* GL_2(R) & \xleftarrow{\quad f_2^* \quad} H^* X_2(R)
\end{array}
$$

where we have identified $H^* GL_2(R)$ with $H^* BGL_2(R)$. According to Theorem 2.1, Corollary 2.2, and their proofs, there are identifications

$$
H^* D_1(R) \overset{\theta_0}{\approx} \mathbb{F}_p[x] \otimes \bigotimes_{i=1}^{r} \Lambda(y_i),
$$

$$
H^* D_2(R) \approx \mathbb{F}_p[x_1, x_2] \otimes \bigotimes_{i=1}^{r} \Lambda(y_{i,1}, y_{i,2}),
$$

$$
H^* X_2(R) \approx \mathbb{F}_p[c_1, c_2] \otimes \bigotimes_{i=1}^{r} \Lambda(e_{i,1}, e_{i,2})
$$

such that $x = \beta(y_i)$, $x_1 = \beta(y_{i,1})$, $x_2 = \beta(y_{i,2})$, $c_1 = \beta(e_{i,1})$, $2c_2 = \beta(e_{i,2})$, $res_2 \circ f_2^*$ is the inclusion determined by $e_{i,1} = y_{i,1} + y_{i,2}$, $e_{i,2} = x_1 y_{i,2} + x_2 y_{i,1}$, and τ^* is the map determined by $\tau^*(y_{i,1}) = 1 \otimes y_i - y_i \otimes 1$, $\tau^*(y_{i,2}) = y_i \otimes 1 + 1 \otimes y_i$ for all $1 \le i \le r$. Assuming these identifications and notation, we have the following lemmas which imply Proposition 4.3:

Lemma 4.5. *The map τ^* is a ring isomorphism mapping the image M_2 of $res_2 \circ f_2^*$ to the following subring inside $H^* D_1(R) \otimes H^* D_1(R)$:*

$$\tau^*(M_2) = \mathbb{F}_p[x^2 \otimes 1, 1 \otimes x] \otimes \bigotimes_{i=1}^{r} \Lambda(xy_i \otimes 1, 1 \otimes y_i).$$

Proof. By applying β to $\tau^*(y_{i,1})$ and $\tau^*(y_{i,2})$ for any $1 \le i \le r$, we obtain $\tau^*(x_1) = 1 \otimes x - x \otimes 1$ and $\tau^*(x_2) = 1 \otimes x + x \otimes 1$. Hence, we easily see that τ^* is surjective and therefore, an isomorphism, because 2 is invertible (p is odd) and the source and the target of τ^* are graded vector spaces of the same finite dimension in each degree. Also, $\tau^*(e_{i,1}) = 2(1 \otimes y_i)$, $\tau^*(e_{i,2}) = 2(1 \otimes xy_i - xy_i \otimes 1)$ for $1 \le i \le r$ and by applying β, $\tau^*(c_1) = 2(1 \otimes x)$, $\tau^*(c_2) = 2(1 \otimes x^2 - x^2 \otimes 1)$. Therefore, $1 \otimes y_i$, $1 \otimes x$, $xy_i \otimes 1$, and $x^2 \otimes 1$ for $1 \le i \le r$ generate $\tau^*(M_2)$ as a ring. $\square$

Lemma 4.6. *The image of ι^* is invariant with respect to the ring automorphism of $H^* D_1(R)$ given by $x \mapsto -x$, $y \mapsto -y_i$ for $1 \le i \le r$.*

Proof. The group automorphism of $D_1(R)$ given by $a \mapsto a^{-1}$ for $a \in D_1(R)$ induces the multiplication by (-1) map on $H^1 D_1(R)$. The map $\iota(a) \to \iota(a^{-1})$ for $a \in D_1(R)$ is a conjugation inside $SL_2(R)$ which induces the trivial map on $H^* SL_2(R)$. Hence, we conclude that the image of ι^* is invariant with respect to the map described in the lemma. $\square$

Lemma 4.7. *If all étale obstruction classes associated to θ_0 in $H_* SL_2(R)$ vanish, then $\tau^*(I_2) = \tau^*(M_2)$.*

Proof. The idea is to use the additive basis of $H^* D_1(R) \otimes H^* D_1(R)$ given by the monomials $x^I \otimes x^J$ with $I, J \in S(1)$. By Lemma 4.5, the monomials $x^I \otimes x^J$ where $I, J \in S(1)$ with $a(I)$ a non-positive integer form an additive basis for $\tau^*(M_2)$. By Lemma 4.6, the monomials $x^I \otimes x^J$ where $I, J \in S(1)$ with $a(I)$ an integer span a linear subspace that contains the image of $\iota^* \otimes Id$. Finally, by our hypothesis, given any $z \in H^* SL_2(R)$, $z' \in H^* D_1(R)$, and $I, J \in S(1)$ with $a(I)$ a positive integer, the following equation is satisfied:

$$\langle \iota^*(z) \otimes z', u_I \otimes u_J \rangle = \langle z, \iota_*(u_I) \rangle \langle z', u_J \rangle = 0.$$

Here, (u_L) denotes the dual basis in $H_* D_1(R)$ of the basis (x^L) in $H^* D_1(R)$ where $L \in S(1)$ and $\langle\,,\,\rangle$ is the duality pairing. Hence, by putting together these observations we conclude that $Im(\iota^* \otimes Id) \subseteq \tau^*(M_2)$. According to the diagram (4.3), we have $\tau^*(M_2) \subseteq \tau^*(I_2) \subseteq Im(\iota^* \otimes Id)$ and the conclusion follows. $\square$

Lemma 4.8. *If there is an étale obstruction class associated to θ_0 in $H_* SL_2(R)$ that is not zero, then $\tau^*(I_2) \ne \tau^*(M_2)$.*

Proof. By a spectral sequence argument, $Im(\pi^*)$ agrees with the invariants of $H^* SL_2(R) \otimes H^* D_1(R)$ with respect to the action of the cokernel of π, which is $D_1(R)/D_1(R)^2 \approx (\mathbb{Z}/2)^r$. Because this group acts trivially on $H^* D_1(R) \otimes H^* D_1(R)$ and 2 is invertible, we conclude that

(4.4) $Im(\iota^* \otimes Id) = (\iota^* \otimes Id)(Im(\pi^*)) = \tau^*(I_2).$

With the same notation as in the proof of Lemma 4.7, if there is $I_0 \in S(1)$ with $a(I_0)$ a positive integer and $\iota_*(u_{I_0}) \ne 0$, then there is $z \in H^* SL_2(R)$ such that $\langle z, \iota_*(u_{I_0}) \rangle \ne 0$. Consequently, $\iota^*(z) \otimes 1$ belongs to $\tau^*(I_2)$ by (4.4) but not to $\tau^*(M_2)$ by Lemma 4.5. $\square$

Lemma 4.9. *The linear subspace of $H_*SL_2(R)$ spanned by the étale obstruction classes is independent of the admissible identification chosen.*

Proof. Let $(x; y_1, y_2, ..., y_r)$ and $(\bar{x}; \bar{y}_1, \bar{y}_2, ..., \bar{y}_r)$ be two sets of ring generators for $H^*D_1(R)$ with $x, \bar{x} \in \beta H^1 D_1(R)$ and $y_i, \bar{y}_i \in H^1 D_1(R)$ for $1 \leq i \leq r$. Because $\beta H^1 D_1(R)$ is a one-dimensional vector space, $\bar{x} = \lambda x$ for some nonzero scalar $\lambda \in \mathbb{F}_p$. Because $H^1 D_1(R)$ is r-dimensional, there is $(\theta_{i,j}) \in GL_r(\mathbb{F}_p)$ such that

$$\bar{y}_i = \sum_{j=1}^{r} \theta_{i,j} y_j \quad \text{for } 1 \leq i \leq r.$$

Then, for each $I \in S(1)$ of the form $I = (k; \epsilon_1, \epsilon_2, ..., \epsilon_r)$ the monomial

$$\bar{x}^I = \bar{x}^k \prod_{i=1}^{r} \bar{y}_i^{\epsilon_i}$$

can be written as a linear combination of monomials of the form x^{I_σ} where $\sigma \in \Sigma_r$ and $I_\sigma = (k; \epsilon_{\sigma(1)}, \epsilon_{\sigma(2)}, ..., \epsilon_{\sigma(r)})$ and vice versa. Observe that $a(I) = a(I_\sigma)$ for any $I \in S(1)$ and $\sigma \in \Sigma_r$. Let (u_I) be the dual basis of (x^I) and $(\bar{u}_I)$ the dual of $(\bar{x}^I)$ where $I \in S(1)$. Hence, we conclude that each étale obstruction class $\iota_*(u_J)$ with $J \in S(1)$ is a linear combination of $\iota_*(\bar{u}_I)$ where $I \in S(1)$ with $a(I)$ a positive integer, and vice versa. $\square$

4.2. More general rings of integers.
According to [2], we can generalize Theorem 1.1 as follows. Suppose that F is a finite field extension of $\mathbb{Q}$ and that $\mathcal{O}_F$ is the ring of algebraic integers in F. Assume that $\mathcal{O}_F$ satisfies the following conditions:

(1) in $\mathcal{O}_F$ there is only one prime ideal above p;
(2) the Picard group of $\mathcal{O}_F$ has no p-torsion;
(3) $\mathcal{O}_F$ contains a primitive p-th root of unity.

Let $R = \mathcal{O}_F[1/p]$ and $r = r_2 + 1$ where r_2 is half the number of complex embeddings of F. Then we can define the *étale obstruction classes* associated to an admissible identification of $H^*D_1(R)$ with a ring of the form (4.2) exactly as in Definition 4.2. With these adjustments, the proofs go through word by word and we have the following overall result:

Theorem 4.10. *Let $R = \mathcal{O}_F[1/p]$ with p an odd prime and $\mathcal{O}_F$ a ring of algebraic integers satisfying (1) to (3) above. Then the linear subspace of $H_*SL_2(R)$ spanned by the étale obstruction classes associated to an admissible identification is trivial if and only if the maps res_n and $res_n \circ f_n^*$ as in Theorem 1.1 have the same image for all $n \geq 0$, including in the stable range.*

Remark 4.11. The statement that res_n and $res_n \circ f_n^*$ have the same image in the stable range was proved in a paper of S. Mitchell [7]. It should be noted that $H^*BD_n(R)$ and $H^*X_n(R)$ are completely stable in the sense that they are images of the $n = \infty$ versions, but this is not true for $H^*BGL_n(R)$. Hence the case $n < \infty$ does not follow from Mitchell's result.

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